Digital Image Processing and Pattern Recognition

E1528

Fall 2021-2022

Lecture 11

Two-Dimensional Discrete Fourier Transform

INSTRUCTOR

DR / AYMAN SOLIMAN

Contents

- Equations Summary
- 2D DFT and Inverse DFT
- Computation of 2D-DFT
- Convolution Theorem
- 2D-DFT Domain Filter Design

➢ Summary of FT, FS, DTFT/DSFT, DFS, DFT and FFT

Fourier Transform (FT):

$$\begin{tikzcd}[column sep=huge] x(t) \arrow[r, shift left, "X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt"] & X(\Omega) \arrow[l, shift left, "x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega"] \end{tikzcd}$$

(continuous) (continuous)

Fourier Series (FS):

$$\begin{tikzcd}[column sep=huge] x(t) \arrow[r, shift left, "X_m = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jm\Omega_0 t}dt"] & X_m \arrow[l, shift left, "x(t) = \sum_{m=-\infty}^{\infty} X_m e^{jm\Omega_0 t}"] \end{tikzcd}$$

$$\Omega_0 = \frac{2\pi}{T}$$

(continuous, periodic) (discrete)

Discrete Time/Space Fourier Transform (DTFT/DSFT):

$$\begin{tikzcd}[column sep=huge] x(n) \arrow[r, shift left, "X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}"] & X(\omega) \arrow[l, shift left, "x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega"] \end{tikzcd}$$

(discrete) (continuous, periodic)

➢ Summary of FT, FS, DTFT/DSFT, DFS, DFT and FFT

Discrete Fourier Series (DFS):

$$\tilde{x}(n) \quad \begin{array}{c} \tilde{X}_m = \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j2\pi nm/N} \\ \longrightarrow \\ \longleftarrow \\ \tilde{x}(n) = \frac{1}{N} \sum_{m=0}^{N-1} \tilde{X}_m e^{j2\pi mn/N} \end{array} \quad \tilde{X}_m$$

(discrete, periodic)

(discrete, periodic)

Discrete Fourier Transform (DFT):

$$x(n) \quad \begin{array}{c} X(m) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nm/N} \\ \longrightarrow \\ \longleftarrow \\ x(n) = \frac{1}{N} \sum_{m=0}^{N-1} X(m) e^{j2\pi mn/N} \end{array} \quad X(m)$$

(discrete, finite)

(discrete, finite)

Fast Fourier Transform (FFT): Fast algorithm for computing DFT

Two-Dimensional Discrete Fourier Transform (2D-DFT)

➢ 2D-DFT and Inverse DFT (IDFT)

$$F(u, v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

$$\begin{tikzcd} f(x, y) \arrow[r, shift left] & F(u, v) \arrow[l, shift left] \end{tikzcd}$$

$$f(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$

M, *N*: image size

x, *y*: image pixel position

u, *v*: spatial frequency

often used short notation:

$$W_N = e^{-j2\pi/N}$$

➢ The Meaning of DFT and Spatial Frequencies

- **Important Concept**

  Any signal can be represented as a linear combination of a set of basic components

$$f(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$

  - Fourier components: sinusoidal patterns
  - Fourier coefficients: weighting factors assigned to the Fourier components

- Spatial frequency: The frequency of Fourier component
- Not to confused with electromagnetic frequencies (e.g., the frequencies associated with light colors)

➢ Real Part, Imaginary Part, Magnitude, Phase, Spectrum

Real part: $R = \text{Real}(F)$

Imaginary part: $I = \text{Imag}(F)$

Magnitude-phase representation: $F(u, v) = |F(u, v)|e^{-j\phi(u,v)}$

Magnitude (spectrum): $|F(u, v)| = \left[R^2(u, v) + I^2(u, v)\right]^{1/2}$

Phase (spectrum): $\phi(u, v) = \tan^{-1}\left[\frac{I(u, v)}{R(u, v)}\right]$

Power Spectrum: $P(u, v) = |F(u, v)|^2$

➢ 2D DFT Properties

Mean of image/ DC component: $\bar{f}(x, y) = F(0, 0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x, y)$

Highest frequency component: $f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$

“Half-shifted” Image: $f(x - M/2, y - N/2) \Leftrightarrow F(u, v)(-1)^{u+v}$

Conjugate Symmetry: $F(u, v) = F^{*}(-u, -v)$

Magnitude Symmetry: $|F(u, v)| = |F(-u, -v)|$

➢ 2D DFT Properties

Spatial domain differentiation: $$\frac{\partial^n f(x, y)}{\partial x^n} \Leftrightarrow (ju)^n F(u, v)$$

Frequency domain differentiation: $$(-jx)^n f(x, y) \Leftrightarrow \frac{\partial^n F(u, v)}{\partial u^n}$$

Distribution law: $$\Im[f_1(x, y) + f_2(x, y)] = \Im[f_1(x, y)] + \Im[f_2(x, y)]$$

Laplacian: $$\nabla^2 f(x, y) \Leftrightarrow -(u^2 + v^2)F(u, v)$$

Spatial domain Periodicity: $$f(x, y) = f(x + M, y) = f(x, y + N) = f(x + M, y + N)$$

Frequency domain periodicity: $$F(u, v) = F(u + M, v) = F(u, v + N) = F(u + M, v + N)$$

➢ Computation of 2D-DFT

Fourier transform matrices: remember $W_N = e^{-j2\pi/N}$

$$\mathrm{F}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix} \qquad \mathrm{F}_N^* = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{1-N} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & W_N^{1-N} & W_N^{2(1-N)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}$$

relationship: $\mathrm{F}_N^{-1} = \frac{1}{N}\mathrm{F}_N^*$

In particular, for $N = 4$:

$$\mathrm{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad \mathrm{F}_4^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

➢ Computation of 2D-DFT

- To compute the 1D-DFT of a 1D signal **x** (as a vector):

$$\tilde{\mathrm{x}} = \mathrm{F}_N \mathrm{x}$$

To compute the inverse 1D-DFT:

$$\mathrm{x} = \frac{1}{N} \mathrm{F}_N^* \tilde{\mathrm{x}}$$

- To compute the 2D-DFT of an image **X** (as a matrix):

$$\widetilde{\mathrm{X}} = \mathrm{F}_N \mathrm{X} \mathrm{F}_N$$

To compute the inverse 2D-DFT:

$$\mathrm{X} = \frac{1}{N^2} \mathrm{F}_N^* \widetilde{\mathrm{X}} \mathrm{F}_N^*$$

➢ Computation of 2D-DFT: Example

- A 4x4 image
- Compute its 2D-DFT:

$$\mathrm{X} = \begin{bmatrix} 1 & 3 & 6 & 8 \\ 9 & 8 & 8 & 2 \\ 5 & 4 & 2 & 3 \\ 6 & 6 & 3 & 3 \end{bmatrix}$$

$$\widetilde{\mathrm{X}} = \mathrm{F}_4\mathrm{X}\mathrm{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 & 3 & 6 & 8 \\ 9 & 8 & 8 & 2 \\ 5 & 4 & 2 & 3 \\ 6 & 6 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

MATLAB function: *fft2*

$$= \begin{bmatrix} 21 & 21 & 19 & 16 \\ -4-3j & -1-2j & 4-5j & 5+j \\ -9 & -7 & -3 & 6 \\ -4+3j & -1+2j & 4+5j & 5-j \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

lowest frequency component

$$= \begin{bmatrix} 77 & 2-5j & 3 & 2+5j \\ 4-9j & -11+8j & -4-7j & -5-4j \\ -13 & -6+13j & -11 & -6-13j \\ 4+9j & -5+4j & -4+7j & -11-8j \end{bmatrix}$$

highest frequency component

➢ Computation of 2D-DFT: Example

$$\tilde{X} = \begin{bmatrix} 77 & 2-5j & 3 & 2+5j \\ 4-9j & -11+8j & -4-7j & -5-4j \\ -13 & -6+13j & -11 & -6-13j \\ 4+9j & -5+4j & -4+7j & -11-8j \end{bmatrix}$$

Real part:

$$\tilde{X}_{real} = \begin{bmatrix} 77 & 2 & 3 & 2 \\ 4 & -11 & -4 & -5 \\ -13 & -6 & -11 & -6 \\ 4 & -5 & -4 & -11 \end{bmatrix}$$

Imaginary part:

$$\tilde{X}_{imag} = \begin{bmatrix} 0 & -5 & 0 & 5 \\ -9 & 8 & -7 & -4 \\ 0 & 13 & 0 & -13 \\ 9 & 4 & 7 & -8 \end{bmatrix}$$

Magnitude:

$$\tilde{X}_{magnitude} = \begin{bmatrix} 77 & 5.39 & 3 & 5.39 \\ 9.85 & 13.60 & 8.06 & 6.4 \\ 13 & 14.32 & 11 & 14.32 \\ 9.85 & 6.40 & 8.06 & 13.60 \end{bmatrix}$$

Phase:

$$\tilde{X}_{phase} = \begin{bmatrix} 0 & -1.19 & 0 & 1.19 \\ -1.15 & 2.51 & -2.09 & -2.47 \\ 3.14 & 2.00 & 3.14 & -2.00 \\ 1.15 & 2.47 & 2.09 & -2.51 \end{bmatrix}$$

➢ Computation of 2D-DFT: Example

- Compute the inverse 2D-DFT:

$$\mathrm{F}_4^*\widetilde{\mathrm{X}}\mathrm{F}_4^* = \frac{1}{4^2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{bmatrix}\begin{bmatrix}77 & 2-5j & 3 & 2+5j\\ 4-9j & -11+8j & -4-7j & -5-4j\\ -13 & -6+13j & -11 & -6-13j\\ 4+9j & -5+4j & -4+7j & -11-8j\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{bmatrix}\begin{bmatrix}21 & 21 & 19 & 16\\ -4-3j & -1-2j & 4-5j & 5+j\\ -9 & -7 & -3 & 6\\ -4+3j & -1+2j & 4+5j & 5-j\end{bmatrix}$$

$$= \begin{bmatrix}1 & 3 & 6 & 8\\ 9 & 8 & 8 & 2\\ 5 & 4 & 2 & 3\\ 6 & 6 & 3 & 3\end{bmatrix} = \mathrm{X}$$

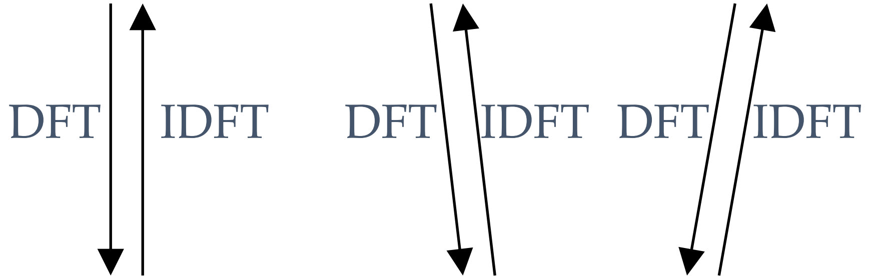
MATLAB function: *ifft2*

2D-DFT (Frequency) Domain Filtering

➢ Convolution Theorem

$f(x,y)$	$h(x,y)$	$g(x,y)$
input image	impulse response (filter)	output image

$$g(x,y) = f(x,y) \otimes h(x,y)$$



$$G(u,v) \;=\; F(u,v)\,H(u,v)$$

➢ Frequency Domain Filtering

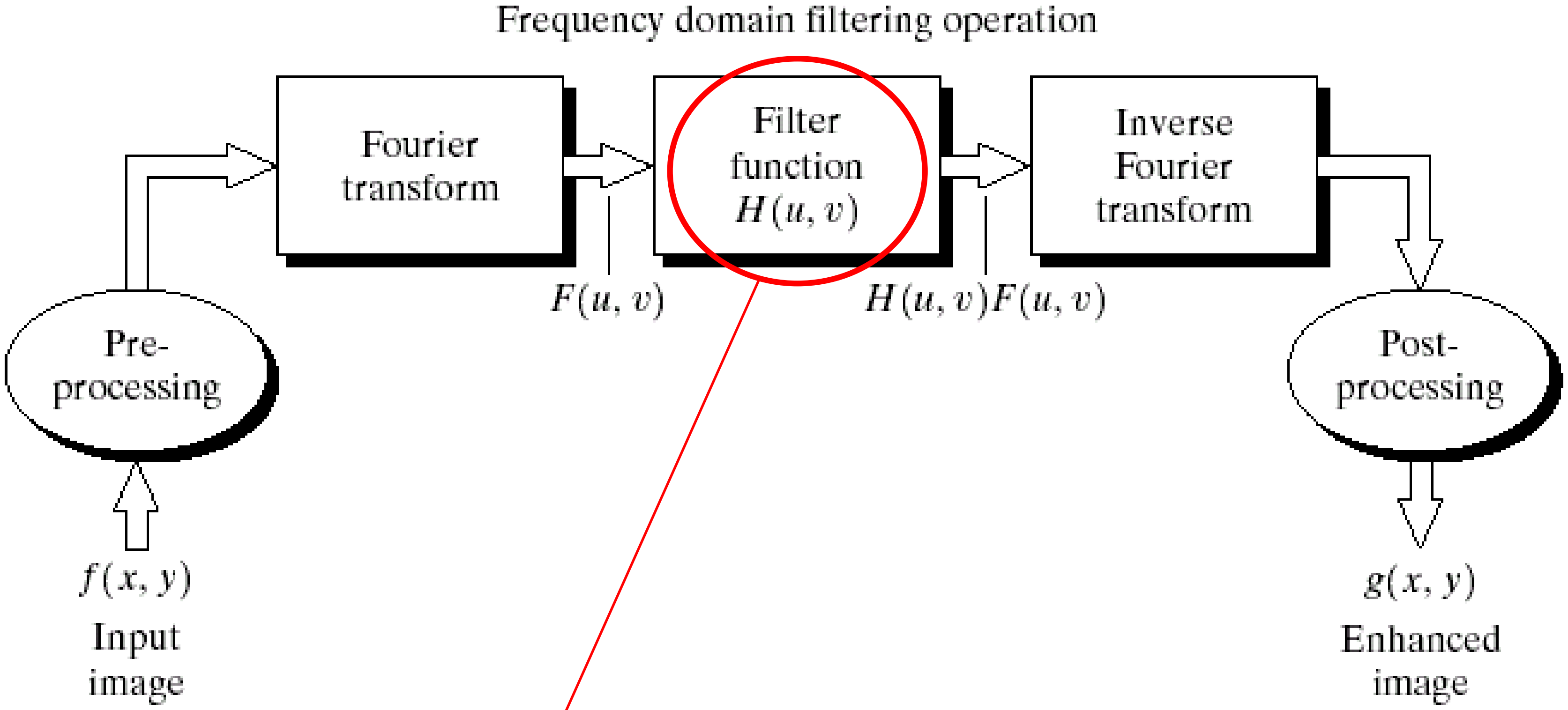


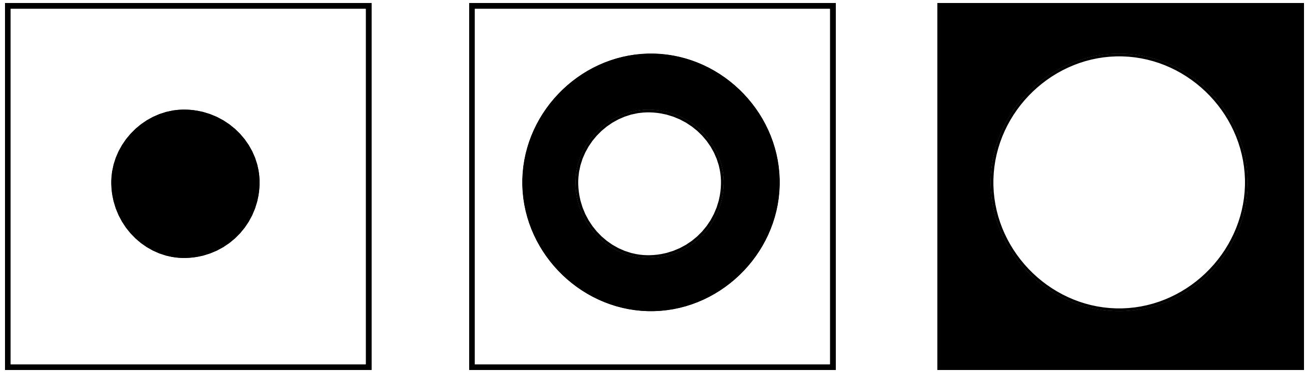
FIGURE 4.5 Basic steps for filtering in the frequency domain.

Filter design: design $H(u,v)$

From [Gonzalez & Woods]

➢ 2D-DFT Domain Filter Design

- Ideal lowpass, bandpass and highpass



low-frequency mask

mid-frequency mask

high-frequency mask

➢ 2D-DFT Domain Filter Design

- Ideal lowpass, bandpass and highpass

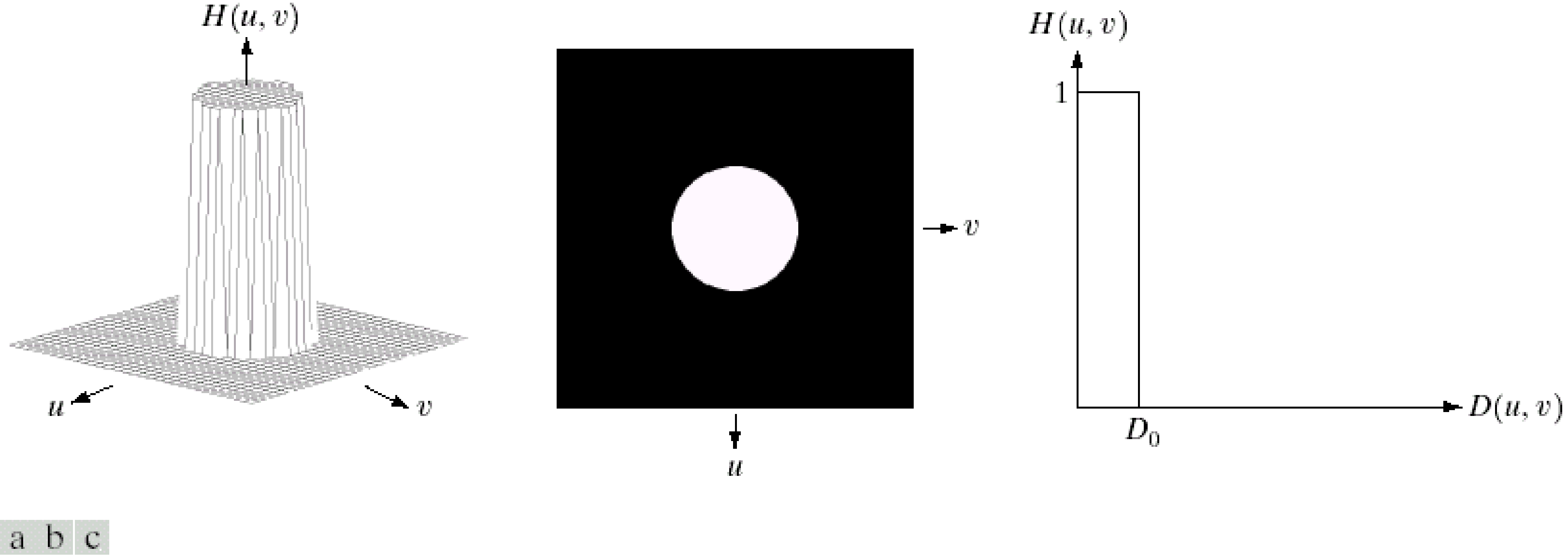


FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

2D-DFT Domain Filter Design

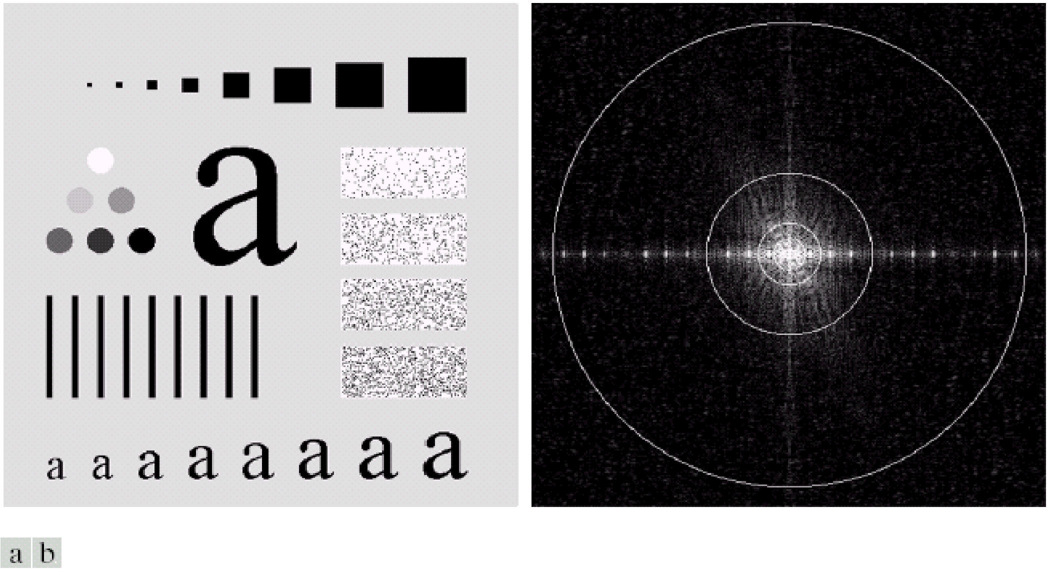
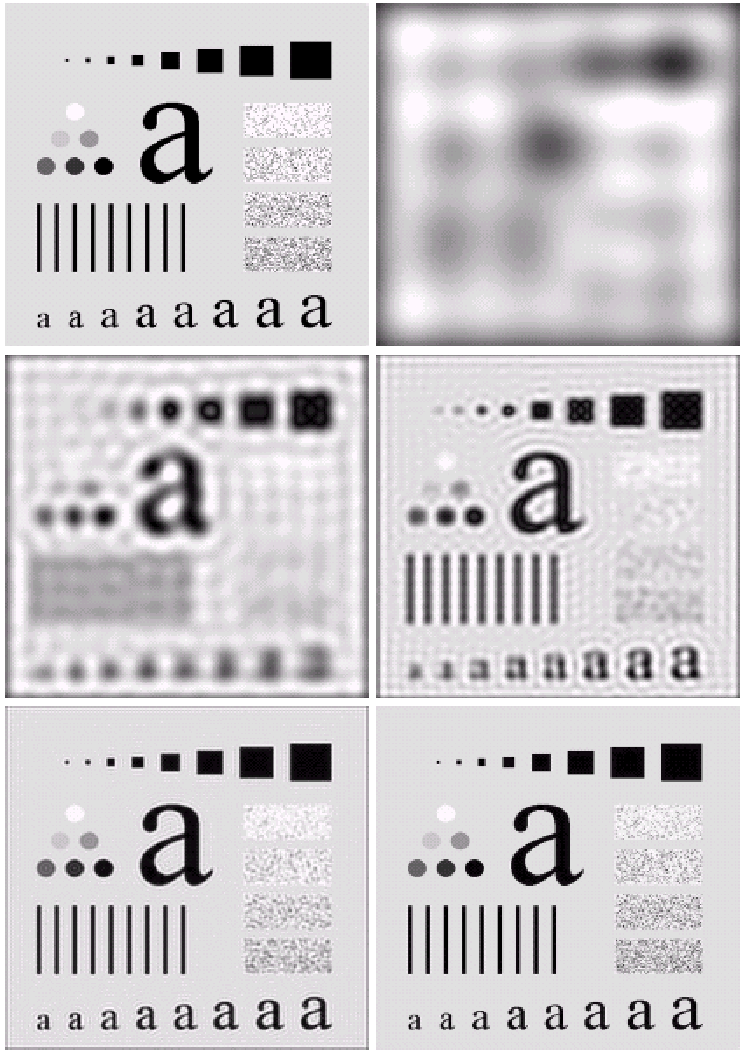


FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

Ideal lowpass filtering with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, respectively



From [Gonzalez & Woods]

➢ 2D-DFT Domain Filter Design

- Gaussian lowpass

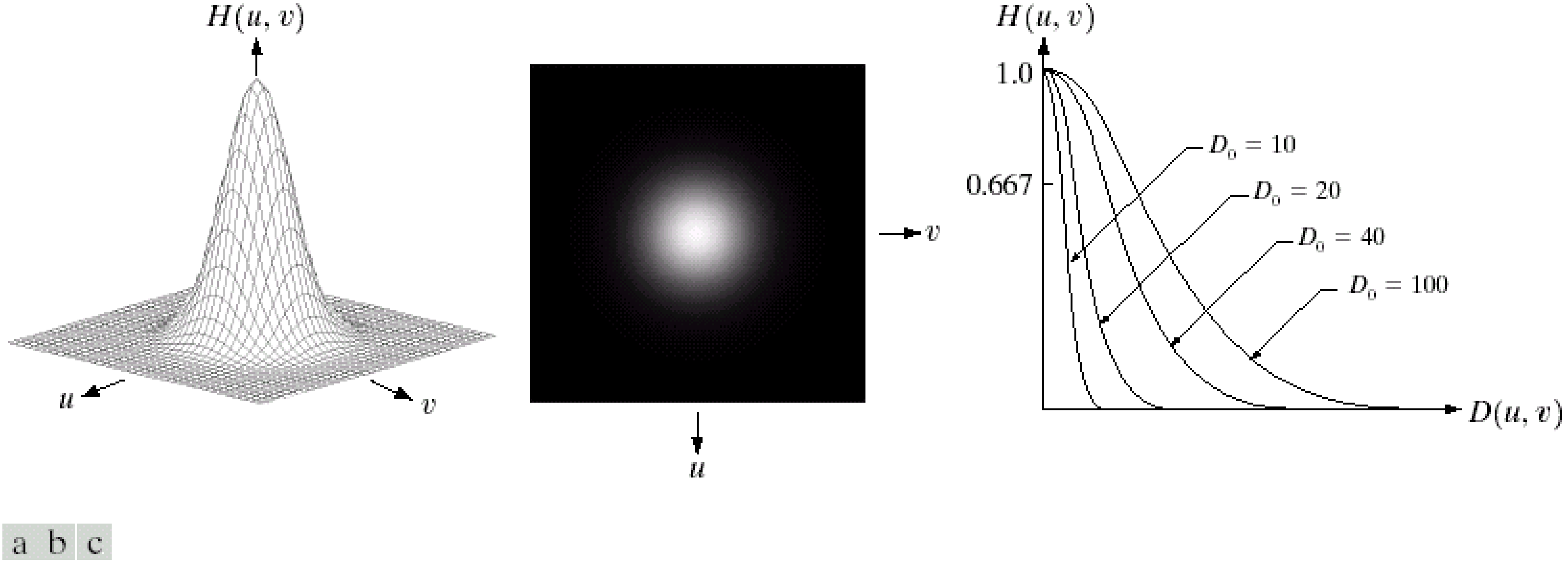
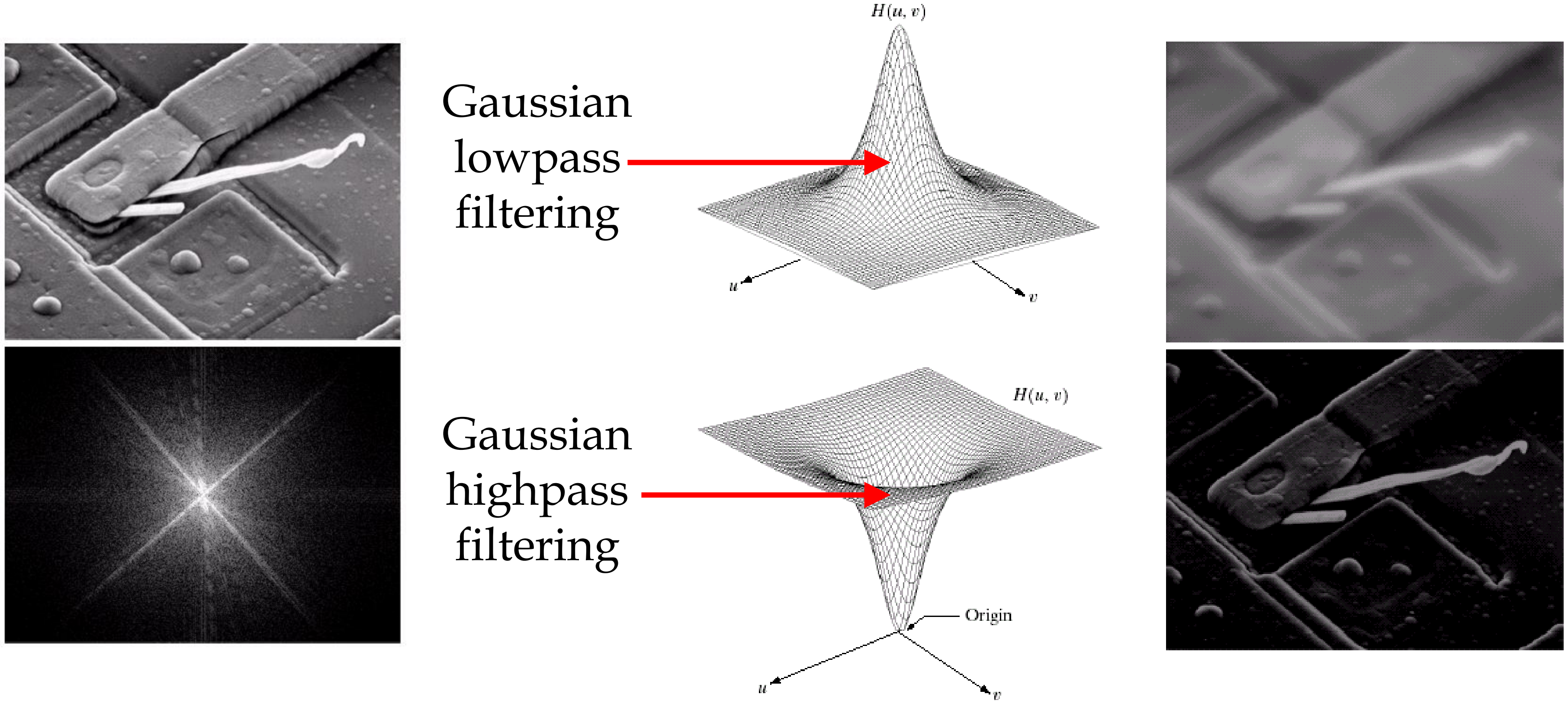


FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

From [Gonzalez & Woods]

➢ 2D-DFT Domain Filter Design



Gaussian lowpass filtering

Gaussian highpass filtering

From [Gonzalez & Woods]

FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

2D-DFT Domain Filter Design

- Choices of high pass filters

Ideal Butterworth Gaussian

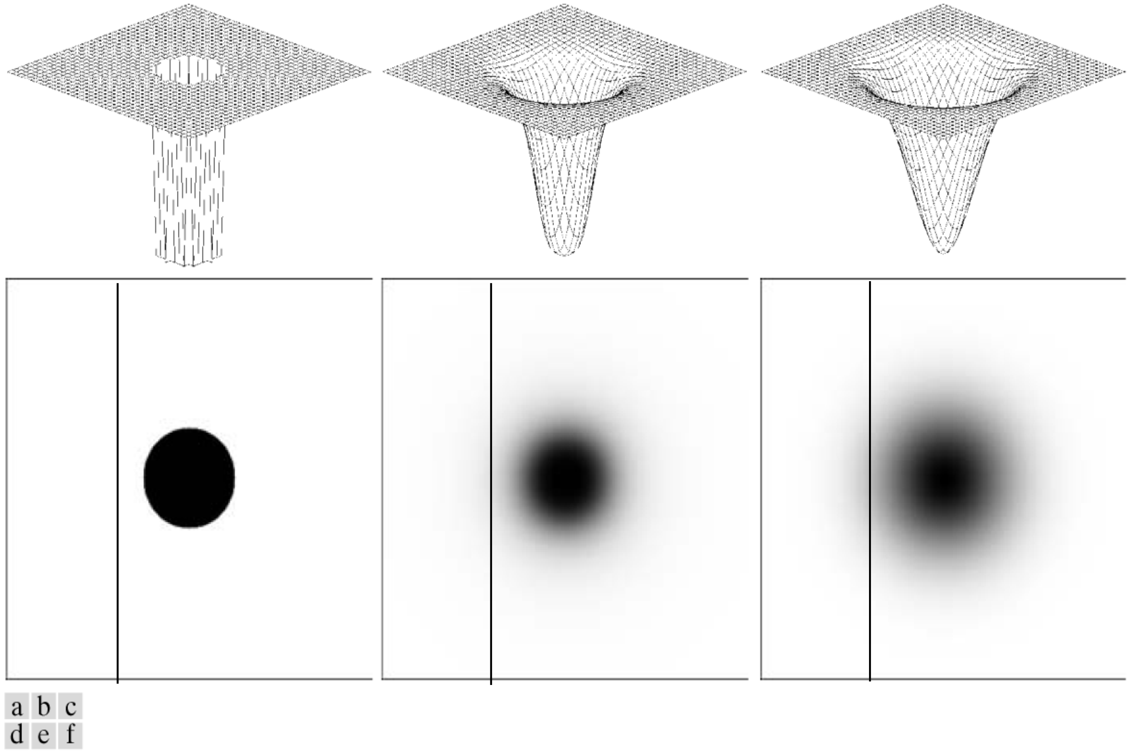
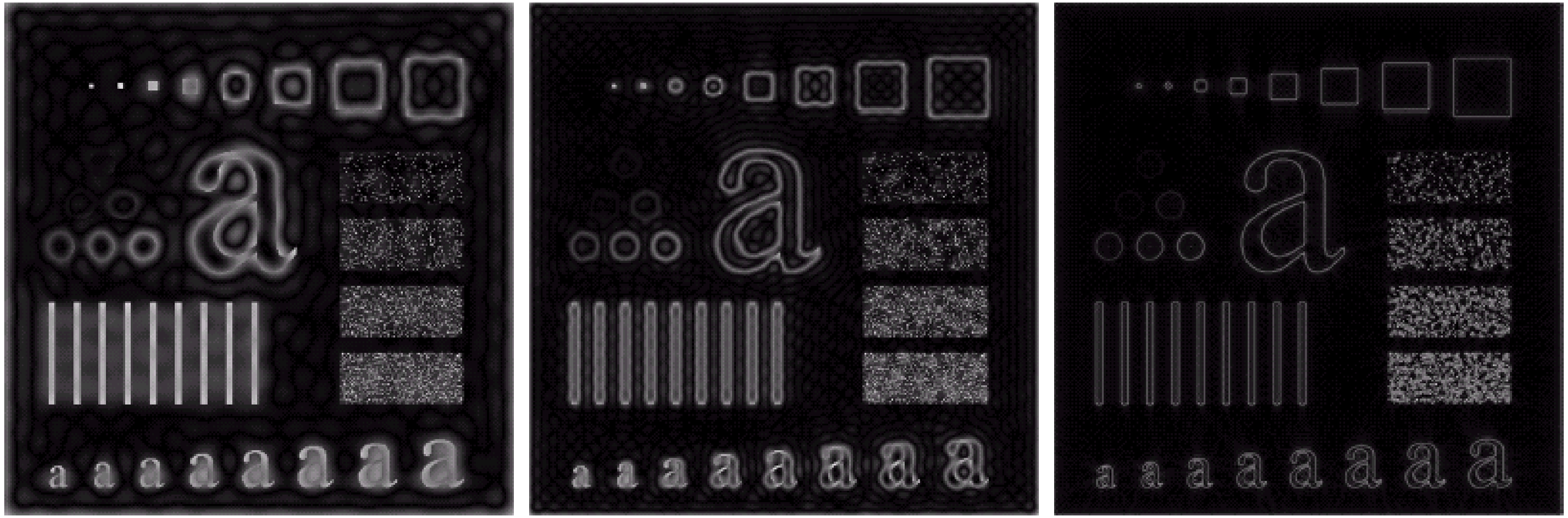


FIGURE 4.17 Top row: Perspective plots of ideal, Butterworth, and Gaussian highpass filters. Bottom row: Corresponding images.

From [Gonzalez & Woods]

➢ 2D-DFT Domain Filter Design

Ideal Butterworth Gaussian



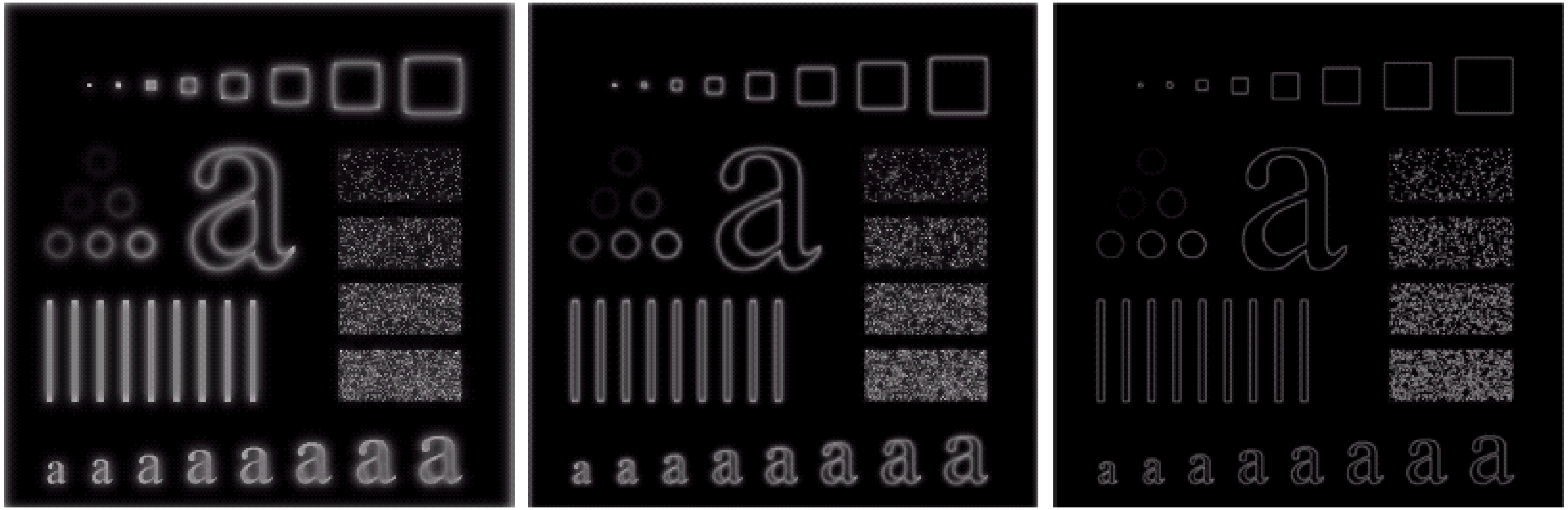
a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

From [Gonzalez & Woods]

2D-DFT Domain Filter Design

Gaussian filter with different width



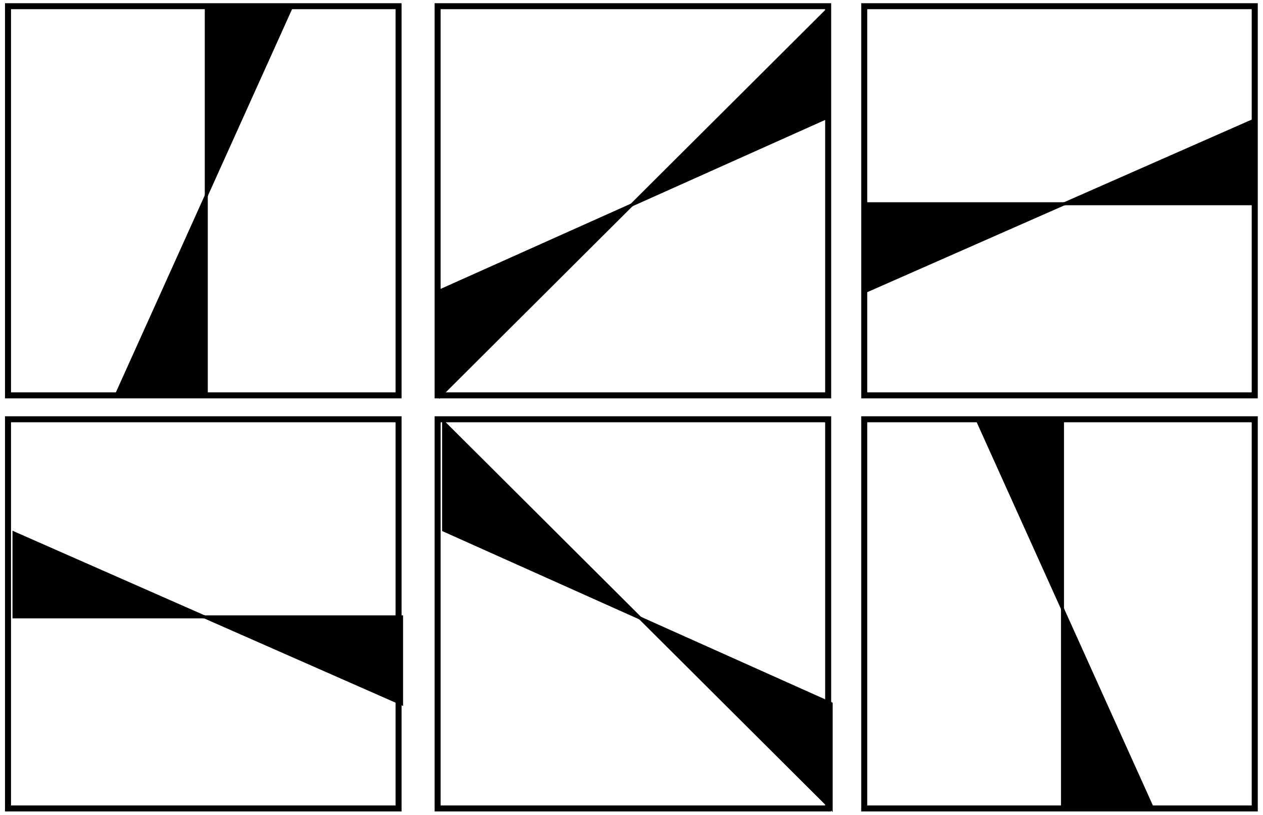
a b c

FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.

From [Gonzalez & Woods]

2D-DFT Domain Filter Design

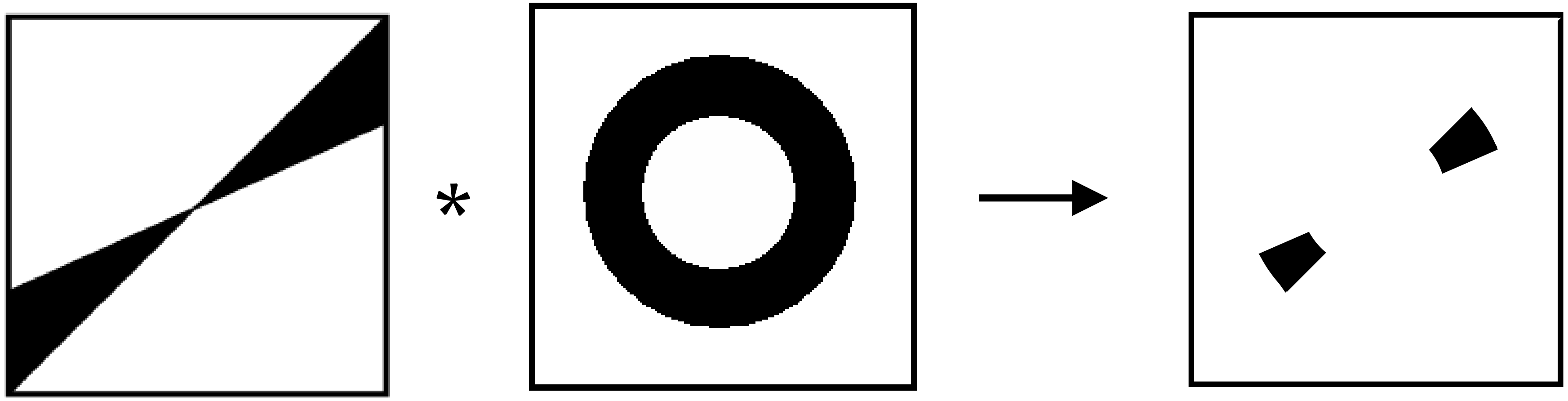
- Orientation selective filters



➢ 2D-DFT Domain Filter Design

- Narrowband Filtering

by combining radial and orientation selection



Thank you